Kinetic equations for a dissipative quantum system driven by dichotomous noise: An exact result

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(Received 26 May 1994)

We obtain kinetic equations for the stochastically averaged reduced density matrix of a quantum system driven by a dichotomous Markovian process and weakly coupled to a quantum thermal bath. These kinetic equations are *exact* in arbitrary dichotomous perturbation and provide a unified description of both small and large correlation time limits, along with an intermediate case. As an example, the problem of a dissipative two-level system with a dichotomically modulated difference in eigenenergies is considered.

PACS number(s): 02.50.Ey, 03.65.Bz, 02.50.Wp, 05.60.+w

Quantum-dynamical systems (QDS) with a finite number of states are ubiquitous in physics. Such systems are frequently interacting with an environment which causes the damping and dephasing effects in the QDS dynamics. The vast literature has been devoted to discussing these effects in the different contexts (see, for example, [1] and references therein). So far, two different theoretical approaches have been used to account for the environment influence on the QDS dynamics. One is the dynamical approach. Within this approach, the environment is usually modeled by a set of independent harmonic oscillators being in the thermal equilibrium [thermal bath (TB)]. The dynamical approach is based on the removing of the TB variables from the Liouville equation for the whole system (QDS+TB) by means of an appropriate elimination procedure (using, for example, the projection operator method [2] or the path-integral approach [1]). The main idea of the alternative stochastic approach is to treat the QDS-environment interaction phenomenologically via the introduction of a stochastic term in the QDS Hamiltonian [3]. Both methods have limitations. The dynamical approach is rather tedious to be practical beyond the harmonic approximation for TB. In turn, the pure stochastic approach is unable to get a description valid at finite temperatures [4–6]. Several efforts to overcome these shortcomings in the framework of a combined approach have been recently undertaken [4–8]. The main idea of Refs. [6,7] was to model the environment influence both via the interaction, V, with a quantum TB and through semiclassical stochastic addition, $\tilde{H}(t)$, into the QDS Hamiltonian, H_0. The latter one can be used, for example, to model the highly anharmonic degrees of the TB. This stochastic addition has been handled within the cumulant expansion method [4,9] in [4,6]. However, in such a way it is possible to obtain the averaged kinetic equation only in the lowest approximations over the Kubo number $K = \Delta\tau_c$, which characterizes the strength of fluctuations of the stochastic process $\tilde{H}(t)$ [9]. Here Δ and τ_c are the parameters which characterize the amplitude of fluctuations (hereafter $\hbar = 1$) and autocorrelation time of a stochastic term $\tilde{H}(t)$, respectively.

Recently, an alternative approach to the problem based on the theory of kinetic equations for a QDS in strong external field has been proposed [7,8]. This approach permits one to obtain the averaged kinetic equations in the way which is nonperturbative with respect to the above Kubo number. However, this approach was restricted before to the case of the fast fluctuation when the autocorrelation time of a stochastic perturbation, τ_c, is much shorter than the relaxation time, τ_r, in the quantum system. In the present paper we put forward a way to overcome this restriction in the case of arbitrary dichotomous perturbation $\tilde{H}(t)$. This case is virtually very important because of the fact that it allows us to perform an exact treatment, and the dichotomous noise can be treated as a pre-Gaussian one [10] and hence is used as a simple model for colored noise [11,12].

The main goal of this paper is to obtain the kinetic equation for the averaged reduced density matrix $\langle\rho_{nm}(t)\rangle = \langle n|\langle\rho(t)\rangle|m\rangle$ of QDS. Here $\rho(t) = \mathrm{Tr}\sigma(t)$ is the reduced density operator of the quantum system, $\sigma(t)$ is the density operator of the whole system, $|n\rangle$ is a state of the QDS, Tr denotes the trace over the quantum TB, and $\langle\cdots\rangle$ denotes the average over the stochastic process. With this goal in mind we start from the Argyres and Kelley equation for the reduced density operator of a QDS in a strong external field [13].

Let

$$H(t) = H_0 + \tilde{H}(t) + V + H_T \tag{1}$$

be the Hamiltonian of a whole system, where

$$H_0 = \sum_{nm} H^{(0)}_{nm}\hat{\gamma}_{nm} \tag{2}$$

denotes the Hamiltonian of the QDS written in the basis of the transition operators $\hat{\gamma}_{nm} = |n\rangle\langle m|$,

$$\tilde{H}(t) = \alpha(t)H_1 = \alpha(t)\sum_{nm} H^{(1)}_{nm}\hat{\gamma}_{nm} \tag{3}$$

is the stochastic perturbation, and H_T is the Hamilto-

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nian of a quantum thermal bath. In Eq. (3) $\alpha(t)$ denotes the dichotomous Markov process (DMP) with zero mean and autocorrelation function $\langle\alpha(t+\tau)\alpha(t)\rangle = \exp(-\nu\tau)$ [14]. The operator of quantum system-thermal bath interaction is written in the quite general form as

$$V = \sum_{nm} \hat{F}_{nm}\hat{\gamma}_{nm}, \tag{4}$$

where $\hat{F}_{nm} = \hat{F}^{\dagger}_{mn}$ are the bath-dependent operators with zero average over thermal bath, $\langle\hat{F}_{nm}\rangle_T = \mathrm{Tr}(\rho_T\hat{F}_{nm}) = 0$. The density operator of thermal bath, ρ_T, is supposed to be equilibrium,

$$\rho_T = \exp(-H_T/k_BT)[\mathrm{Tr}\exp(-H_T/k_BT)]^{-1}, \tag{5}$$

where k_B is the Boltzmann constant, and T is the absolute temperature.

Following to Argyres and Kelley [13], we write the reduced density operator equation in the Born approximation in the interaction V and by the initial factorization assumption, $\sigma(0) = \rho(0)\rho_T$, in the form

$$\frac{d}{dt}\rho(t) = -iL(t)\rho(t) - \int_0^t \Gamma(t,t')\rho(t')dt', \tag{6}$$

where

$$L(t) = L_0 + \alpha(t)L_1, \quad L_i = [H_i,(\cdot)], \quad i = 0,1 \tag{7}$$

is stochastic Liouville superoperator, and

$$\Gamma(t,t') = \sum_{kk',rr'} \{K_{rr'kk'}(t-t')[\hat{\gamma}_{rr'}, S(t,t')\hat{\gamma}_{kk'}(\cdot)] - K_{kk'rr'}(t'-t)[\hat{\gamma}_{rr'}, S(t,t')(\cdot)\hat{\gamma}_{kk'}]\} \tag{8}$$

is the memory kernel. In Eq. (8)

$$K_{kk'rr'}(\tau) = \langle\exp(iH_T\tau)\hat{F}_{kk'}\exp(-iH_T\tau)\hat{F}_{rr'}\rangle_T \tag{9}$$

is the bath correlation function, and $S(t,t')$ is the evolution superoperator that fulfills the stochastic evolution equation (SEE) written in the "forward" and "backward" forms as

$$\frac{d}{dt}S(t,t') = -i[L_0 + \alpha(t)L_1]S(t,t'), \quad S(t',t') = I,$$
$$\frac{d}{dt'}S(t,t') = iS(t,t')[L_0 + \alpha(t')L_1], \quad S(t,t) = I. \tag{10}$$

The main peculiarity of Eqs. (6)–(10) is that the dichotomous noise $\tilde{H}(t)$ affects the memory kernel and may be arbitrarily strong.

One must average the master equation (6) over the dichotomous process $\alpha(t)$. With this goal in mind, we proceed as follows. Consider the formal expression

$$\langle\Gamma(t,t'+\tau)\alpha(t'+\tau)\alpha(t')\rho(t')\rangle, \quad \tau > 0. \tag{11}$$

In Eq. (11) the $\Gamma(t,t'+\tau)$ and $\rho(t')$ are functionals of the DMP $\alpha(t)$ involving only times, respectively, posterior to $t'+\tau$ and prior to t'. Therefore, this expression meets the conditions of the Bourret and Frisch theorem (the theorem B in [14]) and can be transformed as

$$\begin{aligned}\langle\Gamma(t,t'+\tau)\alpha(t'+\tau)\alpha(t')\rho(t')\rangle &= \langle\Gamma(t,t'+\tau)\rangle\langle\alpha(t'+\tau)\alpha(t')\rangle\langle\rho(t')\rangle \\ &\quad +\langle\Gamma(t,t'+\tau)\alpha(t'+\tau)\rangle\langle\alpha(t')\rho(t')\rangle.\end{aligned} \tag{12}$$

By passing to the limit $\tau \to +0$ in Eq. (12) we get, using the remarkable property of the DMP, $\alpha^2(t) = 1$, the following corollary of theorem (12)

$$\langle\Gamma(t,t')\rho(t')\rangle = \Gamma^{(0)}(t-t')\rho_0(t') + \Gamma^{(1)}(t-t')\rho_1(t'), \tag{13}$$

where $\Gamma^{(0)}(t-t') = \langle\Gamma(t,t')\rangle$, $\Gamma^{(1)}(t-t') = \langle\Gamma(t,t')\alpha(t')\rangle$, $\rho_0(t) = \langle\rho(t)\rangle$, and $\rho_1(t) = \langle\alpha(t)\rho(t)\rangle$. In the same way we obtain

$$\langle\alpha(t)\Gamma(t,t')\rho(t')\rangle = \Gamma^{(2)}(t-t')\rho_0(t') + \Gamma^{(3)}(t-t')\rho_1(t'), \tag{14}$$

where $\Gamma^{(2)}(t-t') = \langle\alpha(t)\Gamma(t,t')\rangle$ and $\Gamma^{(3)}(t-t') = \langle\alpha(t)\Gamma(t,t')\alpha(t')\rangle$.

To get the equation for the correlator $\rho_1(t)$, one can use the Shapiro and Loginov theorem [15,9]. According to this theorem, any functional, $f(t)$, of the dichotomous process $\alpha(t)$ must obey the following equation:

$$\frac{d}{dt}\langle\alpha(t)f(t)\rangle = -\nu\langle\alpha(t)f(t)\rangle + \left\langle\alpha(t)\frac{d}{dt}f(t)\right\rangle. \tag{15}$$

Using Eqs. (13)–(15), we obtain from Eq. (6) the set of coupled equations for the averaged reduced density operator $\rho_0(t)$ and the correlator $\rho_1(t)$,

$$\begin{aligned}\frac{d}{dt}\rho_0(t) &= -iL_0\rho_0(t) - iL_1\rho_1(t) - \int_0^t \{\Gamma^{(0)}(t-t')\rho_0(t') + \Gamma^{(1)}(t-t')\rho_1(t')\}dt', \\ \frac{d}{dt}\rho_1(t) &= -(\nu + iL_0)\rho_1(t) - iL_1\rho_0(t) - \int_0^t \{\Gamma^{(2)}(t-t')\rho_0(t') + \Gamma^{(3)}(t-t')\rho_1(t')\}dt',\end{aligned} \tag{16}$$

with initial conditions $\rho_0(0) = \rho_0$ and $\rho_1(0) = 0$. The kernels $\Gamma^{(i)}(t-t')$ in Eq. (16) are specified in a similar way to the kernel $\Gamma(t,t')$, Eq. (8), in which the evolution operator $S(t,t')$ is replaced by the averaged operator $S^{(0)}(t-t') = \langle S(t,t')\rangle$, $S^{(1)}(t-t') = \langle S(t,t')\alpha(t')\rangle$, $S^{(2)}(t-t') = \langle\alpha(t)S(t,t')\rangle$, or $S^{(3)}(t-t') = \langle\alpha(t)S(t,t')\alpha(t')\rangle$, respectively. Using the Shapiro and Loginov theorem (15) together with the SEE (10), we find after some algebra the Laplace transforms, $\tilde{S}^{(i)}(p) = \int_0^\infty e^{-p\tau}S^{(i)}(\tau)d\tau$,

$$\tilde{S}^{(0)}(p) = [p + iL_0 + L_1(p+\nu+iL_0)^{-1}L_1]^{-1},$$
$$\tilde{S}^{(1)}(p) = -i\tilde{S}^{(0)}(p)L_1(p+\nu+iL_0)^{-1},$$
$$\tilde{S}^{(2)}(p) = -i(p+\nu+iL_0)^{-1}L_1\tilde{S}^{(0)}(p),$$
$$\tilde{S}^{(3)}(p) = iL_1^{-1}(p+iL_0)\tilde{S}^{(1)}(p) = \tilde{S}^{(2)}(p)(p+iL_0)L_1^{-1}, \tag{17}$$

of the operators $S^{(i)}(\tau)$.

For the averaged density matrix $\rho_{nm}(t)$ and the correlation matrix $\beta_{nm}(t) = \langle n|\rho_1(t)|m\rangle$ one can obtain from Eq. (16) the final set of the coupled kinetic equations,

$$\dot{\rho}_{nm}(t) = -i\sum_{n'm'}\{L^{(0)}_{nmn'm'}\rho_{n'm'}(t) + L^{(1)}_{nmn'm'}\beta_{n'm'}(t)\}$$
$$-\sum_{n'm'}\int_0^t dt'\{\Gamma^{(0)}_{nmn'm'}(t-t')\rho_{n\cdot m'}(t') + \Gamma^{(1)}_{nmn'm'}(t-t')\beta_{n'm'}(t')\}, \tag{18}$$
$$\dot{\beta}_{nm}(t) = -\sum_{n'm'}\{[\nu\delta_{nn'}\delta_{mm'} + iL^{(0)}_{nmn'm'}]\beta_{n'm'}(t) + iL^{(1)}_{nmn'm'}\rho_{n'm'}(t)\}$$
$$-\sum_{n'm'}\int_0^t dt'\{\Gamma^{(2)}_{nmn'm'}(t-t')\rho_{n'm'}(t') + \Gamma^{(3)}_{nmn'm'}(t-t')\beta_{n'm'}(t')\},$$

where

$$L^{(i)}_{nmn'm'} = H^{(i)}_{nn'}\delta_{mm'} - H^{(i)}_{m'm}\delta_{nn'},$$
$$\Gamma^{(i)}_{nmn'm'}(\tau) = \sum_{rr'}\{K_{nrr'n'}(\tau)S^{(i)}_{m'r'mr}(-\tau) + K_{m'r'rm}(-\tau)S^{(i)}_{r'n'rn}(-\tau)$$
$$-K_{m'r'nr}(-\tau)S^{(i)}_{r'n'mr}(-\tau) - K_{rmr'n'}(\tau)S^{(i)}_{m'r'rn}(-\tau)\}, \tag{19}$$

and $S^{(i)}_{nrr'm'}(\tau) = \langle n|S^{(i)}(\tau)\hat{\gamma}_{r'm'}|r\rangle$ are the elements of the Lioville operator L_i, the kernel $\Gamma^{(i)}(\tau)$, and the averaged operator $S^{(i)}(\tau)$ in a supermatrix representation, respectively. The kinetic Eqs. (18), and (19) along with Eq. (17) are the main result of this paper and may be used in a number of applications. It should be particularly emphasized that the derived equations are valid in the limiting cases of both large ($\nu \ll \tau_r^{-1}$) and small ($\nu \gg \tau_r^{-1}$) correlation time, as well as in an intermediate case ($\nu \sim \tau_r^{-1}$).

In order to make this result more concrete, we treat below the simplest illustrative example of a weakly damped two-level system,

$$H_0 = \frac{1}{2}\omega_0(\hat{\gamma}_{11} - \hat{\gamma}_{22}), \tag{20}$$

with dichotomically modulated difference of eigenenergies,

$$H_1 = \frac{1}{2}\varepsilon(\hat{\gamma}_{11} - \hat{\gamma}_{22}). \tag{21}$$

The interaction V with the harmonic TB,

$$H_T = \sum_\lambda \omega_\lambda\left(\hat{b}^+_\lambda\hat{b}_\lambda + \frac{1}{2}\right), \tag{22}$$

is chosen for simplicity in the following form:

$$V = \sum_\lambda \kappa_\lambda(\hat{b}^+_\lambda + \hat{b}_\lambda)[\hat{\gamma}_{12} + \hat{\gamma}_{21}], \tag{23}$$

where ω_λ is the frequency of the λth bath mode, $\hat{b}^+_\lambda$ ($\hat{b}_\lambda$) is the creation (annihilation) operator, and κ_λ is the coupling constant. The simplicity of the considered example lies in the fact that the relevant supermatrices $L_{inmn'm'}$ and $S^{(i)}_{nmn'm'}$ may be rearranged as diagonal 4×4 matrices. Therefore, Eq. (17) is equivalent to a set of independent scalar equations. Besides, the equations for the diagonal and off-diagonal parts of the matrices $\rho_{nm}(t)$, $\beta_{nm}(t)$ are decoupled also.

The bath correlation functions, $K_{nmn'm'}(\tau)$, may be expressed as $K_{nmn'm'}(\tau) = K(\tau)(1-\delta_{nm})(1-\delta_{n'm'})$ in terms of the only one function,

$$K(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega n(\omega)J(\omega)e^{i\omega\tau}, \tag{24}$$

where

$$J(\omega) = 2\pi\sum_\lambda \kappa_\lambda^2[\delta(\omega-\omega_\lambda) - \delta(\omega+\omega_\lambda)] \tag{25}$$

is the TB spectral function, and $n(\omega) = [\exp(\omega/k_BT) - 1]^{-1}$ is the Bose-function. Assume that there exists a correlation time for the TB, τ_0, such that $K(\tau) = 0$ for $|\tau| > \tau_0$ and $\tau_0 \ll \tau_r$. Then on the time scale $t \gg \tau_0$ the upper limit in the integral of Eq. (18) can be replaced by ∞. Using, besides, the Markovian approximation on this time scale, we get ultimately from Eqs. (17)–(19) the pair of coupled differential equations,

$$\dot{n} = -\Gamma_0 n - \Gamma_1 m - \Delta_0,$$
$$\dot{m} = -(\nu+\Gamma_3)m - \Gamma_1 n - \Delta_1, \tag{26}$$

for the populations difference $n(t) = \rho_{11}(t) - \rho_{22}(t)$ and the correlator $m(t) = \langle \alpha(t)n(t)\rangle$. The rate constants Γ_k and constants Δ_k in Eq. (26) are defined as

$$\Gamma_k = \int_{-\infty}^{\infty} d\omega \coth\left(\frac{\omega}{2k_BT}\right) J(\omega)I_k(\omega),$$
$$\Delta_k = \int_{-\infty}^{\infty} d\omega J(\omega)I_k(\omega), \quad (27)$$

where

$$I_k(\omega) = \frac{1}{\pi}\mathrm{Re}[\tilde{S}^{(k)}_{1212}(-i\omega)] \quad (28)$$

is the spectral line shape function. Using Eq. (17), we get for $I_k(\omega)$ the following expressions:

$$I_0(\omega) = \frac{1}{\pi}\frac{\varepsilon^2\nu}{[(\omega-\omega_0)^2-\varepsilon^2]^2+\nu^2(\omega-\omega_0)^2},$$
$$I_1(\omega) = I_2(\omega) = \frac{\omega-\omega_0}{\varepsilon}I_0(\omega),$$
$$I_3(\omega) = \frac{(\omega-\omega_0)^2}{\varepsilon^2}I_0(\omega). \quad (29)$$

To conclude, consider the different limiting cases covered by Eqs. (26)–(29).

(i) Weak noise limit, $K = \varepsilon/\nu \ll 1$. In this case the spectral line shape $I_0(\omega)$ has the sharp Lorentzian form

$$I_0(\omega) = \frac{1}{\pi}\frac{g}{(\omega-\omega_0)^2+g^2}, \quad (30)$$

where $g = \varepsilon^2/\nu$. Because of Eqs. (27)–(30) we have $\Gamma_1 \approx 0$. The latter condition fulfills exactly in the white noise limit: $\varepsilon, \nu \to \infty$, $g =$ const and while $\nu \to \infty$, $\varepsilon =$const or $\varepsilon \to 0$, $\nu =$const. Therefore, we have in these cases a single-exponential decay of the $n(t)$ with the quantum rate Γ_0.

(ii) Strong noise limit, $K = \varepsilon/\nu \gg 1$. In this case

$$I_0(\omega) \approx I_3(\omega) \approx \frac{1}{2}[\delta(\omega-\omega_0+\varepsilon)+\delta(\omega-\omega_0-\varepsilon)],$$
$$I_1(\omega) \approx \frac{1}{2}[\delta(\omega-\omega_0+\varepsilon)-\delta(\omega-\omega_0-\varepsilon)], \quad (31)$$

and thus we have generally the double-exponential decay of the $n(t)$ with the quantum rates

$$\lambda_{1,2} = \frac{\nu}{2} + \Gamma_0 \pm \frac{1}{2}[(\Gamma_+ - \Gamma_-)^2 + \nu^2]^{1/2}, \quad (32)$$

where $\Gamma_\pm = \coth[(\omega_0 \pm \varepsilon)/2k_BT]J(\omega_0 \pm \varepsilon)$ are the quantum rates in the quasistatic limit ($\nu \to 0$) and $\Gamma_0 = (\Gamma_+ + \Gamma_-)/2$ is the average relaxation rate. However, if $\nu \gg \Gamma_0$ (the small correlation time limit), we have $\lambda_1 \approx \nu$, $\lambda_2 \approx \Gamma_0$ and, therefore, the decay of $n(t)$ remains effectively single exponential with the quantum rate Γ_0 in agreement with [7]. Thus Eqs. (26)–(29) reproduce correctly all known limiting cases and provide additional information in an intermediate case ($\varepsilon \sim \nu \sim \tau_r^{-1}$).

I wish to thank Professor E.G. Petrov and Dr. V.I. Teslenko for many fruitful discussions. This work was supported in part by International Science Foundation, Grant No. U4U000.

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